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The motion of a fluid due to a moving source of heat at the boundary

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Experiments performed by Fultz *et al.* (1959) and Stern (1959) indicate that if a flame is rotated around the bottom exterior of a pan or of a circular cylindrical annulus containing water then the fluid acquires a net vertical component of angular momentum in a direction opposite to that of the flame's motion. Stern gave an analysis showing how fluid, bounded by two horizontal plates subjected to small two-dimensional sinusoidal travelling temperature perturbations, could acquire and maintain such momentum. An assumption in Stern's analysis was that the density variations were constant between the plates. Here we shall examine the importance of the relaxation of this assumption. Also, another model, in which the upper surface of the fluid is free, is analysed, because this model conforms more closely with the experiments of Fultz *et al.* and Stern.

For the problem of flow between two plates we shall show that there is a net flux of momentum in the direction opposite to the motion of the thermal field for all frequencies thereof. The same result holds in the case with a free surface when conditions are comparable with those in the experiments.

In both problems the phase lag of the thermal expansion wave with height produces a skewed thermal field. This field is the most important mechanism in producing velocity correlations. It is due to these correlations and the associated Reynolds stress that the fluid acquires its momentum.

1. Introduction

Fultz *et al.* (1959) describe an experiment in which a flame is rotated around the outside bottom rim of a cylindrical vessel filled with water. They found that in the course of the establishment of motion from rest the fluid acquired a net angular momentum in the sense opposite to the motion of the flame. A similar experiment was performed by Stern (1959) who used a cylindrical annulus whose width was small compared with the depth of the water. Thus he was able to reduce the radial convection which was considerable in Fultz's experiments. Stern also observed, by using paper markers and permanganate traces in the fluid, that a net momentum was acquired in the sense opposite to the motion of the flame.

With these experiments in mind Stern examined a two-dimensional model (with no radial drift) to see whether a travelling wave heat source could impart momentum to fluid contained between two horizontal plates. He supposed that each plate was subjected to the same sinusoidal temperature distribution, moving with uniform speed and in the same direction. Stern showed that the net horizontal flux of momentum of the fluid was proportional to the square of the amplitude of the density fluctuations at the plates, and he determined the momentum for small and large values of the frequency. He concentrated on the case when the depth of the fluid is small compared with the horizontal scale of the motion, and we shall also restrict attention to this case.

In Stern's analysis, the density perturbations were taken to be independent of the normal co-ordinate to the plates. This assumption implies that the depth of the fluid is very small relative to a characteristic thermal boundary-layer thickness, based on the speed of the thermal field. The time taken for a thermal conduction wave to propagate vertically through the fluid must therefore be much smaller than the period of the imposed travelling thermal field. This assumption is, however, invalid because the skewed thermal field, rather than viscosity, is the more important factor in creating velocity correlations which produce Reynolds stresses. Stern did have in mind, however, the problem (posed by Halley 1686, but see Stern 1959) where winds in the atmosphere might be produced through periodic radiative heating by the sun.

The limit of the results which we shall obtain, as the Prandtl number tends to zero in the low-frequency case, gives the same value for the net momentum of the flow as Stern obtained (after correcting one of his integrals). At high frequencies, Stern's results are not, unfortunately, a good approximation at small values of the Prandtl number because the asymptotic expansion for the net momentum is not uniformly valid. We shall also discuss the analysis of another model, with the fluid bounded by a plate below and having a free surface above, which would seem to conform more closely to the experiments carried out by Fultz and Stern. One difficulty with this problem lies in the choice of boundary conditions on the thermal field at the free surface. Detailed results will be given only for the case when the gas above the fluid is a perfect insulator. When conditions are comparable with those in the experiments, a net flux of momentum in the opposite direction to the motion of the thermal field also occurs here.

2. Basic analysis

We consider the two-dimensional flow of a viscous fluid of depth h supported below by a horizontal plate. The x-axis is taken to be horizontal at the mid-depth of the fluid and the z-axis is taken as the upward normal to the plate. We suppose that the plate is subjected to a *sinusoidal* temperature field which travels in the *negative* x-direction with uniform speed U. The mean motion, arising from non-linear interactions, is defined to be $\overline{u}(z)$, there is no basic uniform motion. The mean pressure is $\overline{p}(z)$, and u', w', p', ρ' denote the horizontal and vertical velocity fluctuations, the pressure and density fluctuations, respectively, arising from the sinusoidal temperature perturbation. The mean density of the fluid is ρ_0, the kinematic viscosity is ν, and g is the downward vertical component of gravity.

We wish to determine whether the imposed thermal field can produce a horizontal mean motion of the fluid and, if so, to find its average horizontal momentum. The Navier–Stokes and continuity equations are

$$\frac{\partial u'}{\partial t}+(\bar{u}+u')\frac{\partial u'}{\partial x}+w'\frac{\partial}{\partial z}(\bar{u}+u') = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}+\nu\left(\frac{\partial^2\bar{u}}{\partial z^2}+\frac{\partial^2 u'}{\partial x^2}+\frac{\partial^2 u'}{\partial z^2}\right), \tag{1}$$

$$\frac{\partial w'}{\partial t}+(\bar{u}+u')\frac{\partial w'}{\partial x}+w'\frac{\partial w'}{\partial z} = -\frac{1}{\rho_0}\left\{\frac{\partial}{\partial z}(\bar{p}+p')+(\rho_0+\rho')g\right\}+\nu\left(\frac{\partial^2 w'}{\partial x^2}+\frac{\partial^2 w'}{\partial z^2}\right), \tag{2}$$

and

$$\frac{\partial u'}{\partial x}+\frac{\partial w'}{\partial z} = 0. \tag{3}$$

In (1), (2) and (3) we have ignored density variations except for their contribution to the buoyancy force, using the 'Boussinesq approximation'.†

We suppose that ρ'/ρ_0 is small so that we can linearize the perturbation equations and put

$$2u' = \frac{i}{k}\left\{\frac{dw_1}{dz}e^{ik(x+Ut)}-\frac{d\tilde{w}_1}{dz}e^{-ik(x+Ut)}\right\},$$

$$2w' = w_1 e^{ik(x+Ut)}+\tilde{w}_1 e^{-ik(x+Ut)}, \quad 2\rho' = \rho_1 e^{ik(x+Ut)}+\tilde{\rho}_1 e^{-ik(x+Ut)}, \tag{4}$$

where a *tilde* denotes a complex conjugate.

Let us now eliminate p' between the linearized forms of (1) and (2) by differentiating with respect to z and x respectively and subtracting. We may then differentiate the resulting equation with respect to x and use (3) to obtain

$$\left(\frac{\partial}{\partial t}-\nu\frac{\partial^2}{\partial x^2}-\nu\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)w' = -\frac{g}{\rho_0}\frac{\partial^2\rho'}{\partial x^2}. \tag{5}$$

Using (4), the partial differential equation (5) may be written as the ordinary differential equation

$$\left(ikU+\nu k^2-\nu\frac{d^2}{dz^2}\right)\left(k^2-\frac{d^2}{dz^2}\right)w_1 = -\frac{gk^2\rho_1}{\rho_0}. \tag{6}$$

Following Stern, we confine attention to cases where the depth of the fluid is small compared with the wavelength of the thermal field so that $k^2h^2 \ll 4\pi^2$. (Other cases may be treated but the algebra is tortuous.) Making this approximation in (6) we obtain

$$\frac{d^4 w_1}{dz^4}-\frac{ikU}{\nu}\frac{d^2 w_1}{dz^2} = -\frac{gk^2\rho_1}{\nu\rho_0}. \tag{7}$$

When later we give solutions for low values of the frequency $\omega = kU$, we must keep the parameter $(U/k\nu)$ large. This means physically that the characteristic time of the imposed thermal field must be small compared with the horizontal time scale of viscous diffusion.

In the experimental work, heat was transferred by molecular conduction. Hence we believe it to be desirable to calculate the dependence of ρ_1 on z from the temperature equation. We may linearize the temperature equation, an

† The validity of this assumption has been justified *a posteriori*. At low frequencies of the travelling thermal field there is no difficulty. At high frequencies the requirement is that $(k^2h^2)/\Lambda$, (where Λ^{-1} is a Froude number associated with the travelling thermal field), shall be small. This condition is well satisfied for all cases of the two problems considered in this paper.

approximation which involves the neglect of heat convection and dissipation, and obtain

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial z^2}, \tag{8}$$

where T denotes the temperature and κ the thermometric conductivity. We ignore the variation of κ, ν with temperature. Let us write

$$2T = 2\overline{T} + T_1 e^{ik(x+Ut)} + \tilde{T}_1 e^{-ik(x+Ut)}, \tag{9}$$

where $\overline{T}$ is the mean temperature with respect to x. We regard (9) as representing qualitatively the type of heating which occurs in experiments. After all, one can always impose conditions so as to give a realistic distribution at the boundary. As $\overline{T}$ is a function of z alone it follows from (8) and (9) that it must be linear. In the first problem which we shall consider, the value of $\overline{T}$ at the upper boundary is the same as at the lower boundary. In the second problem, the upper surface is supposed to be a good insulator and the depth of the fluid to be small. Thus, in both problems, we may suppose $\overline{T}$ and also the mean density ρ_0 to be constant. Moreover, extracting the fundamental mode from (8) and (9), we have

$$\left(\frac{d^2}{dz^2} - \frac{ikU}{\kappa}\right) T_1 = 0, \tag{10}$$

which, given two boundary conditions on T, enables us to determine T_1. The equation of state tells us that the density variation ρ_1 is proportional to T_1; thus, we may determine the right-hand side of (7) and, consequently, the velocity if given four boundary conditions thereon.

We evaluate the mean motion by extracting the mean part of (1) over one wavelength. With the help of (3) we find that the mean motion, generated by the Reynolds stress, satisfies

$$\nu \frac{d^2\overline{u}}{dz^2} = \frac{d}{dz}(\overline{u'w'}). \tag{11}$$

Equation (11) requires two boundary conditions on $\overline{u}$ for its solution. One necessary condition is that at the plate $\overline{u}$ must be zero. One other condition at the upper surface, such as zero velocity or zero stress, is then sufficient. After having solved (11), we may evaluate the integral of $\overline{u}$ so as to obtain the net momentum of the fluid. We notice that this will be of the order of the square of the mean density fluctuation.

Before proceeding to the next section we complete our definitions with

$$\left.\begin{aligned} z &= h\zeta, & \Omega &= \omega h^2/\kappa, & \alpha^2 &= i\Omega P^{-1}, \\ \omega &= kU, & P &= \nu/\kappa, & \lambda^2 &= i\Omega, \\ & & \Lambda &= gh/U^2. & & \end{aligned}\right\} \tag{12}$$

The quantity Ω is a frequency parameter and measures the ratio of the time scale for heat to diffuse through a distance h, to the time scale imposed by the moving thermal field. Alternatively, we may regard it as the square of the ratio of the height h to a characteristic thermal boundary-layer thickness $(\kappa/\omega)^{\frac{1}{2}}$.

3. The closed problem

In the problem considered by Stern he supposed that the upper surface of the fluid was also bounded by a horizontal plate. The same thermal field was applied to this plate as to the lower plate.

From (10) and (12) the solution for T_1 with boundary conditions $T_1 = T_{1w}$ at $\zeta = \pm\frac{1}{2}$ is

$$\frac{T_1}{T_{1w}} = \frac{\cosh\lambda\zeta}{\cosh\frac{1}{2}\lambda}, \tag{13}$$

where a subscript w denotes the value at the walls. Hence if we use the equation of state with (13), then (7), with the aid of (12), may be written

$$F^{\text{iv}} - \alpha^2 F'' = \alpha^2\lambda^2 \frac{\cosh\lambda\zeta}{\cosh\frac{1}{2}\lambda}, \tag{14}$$

where we have defined $F(\zeta)$ by

$$w_1(z) \equiv \kappa \frac{g\rho_{1w}}{\rho_0 U^2} F(\zeta), \tag{15}$$

and a dash denotes differentiation with respect to ζ. The boundary conditions are $u' = w' = 0$ at the walls so that

$$F = F' = 0 \quad \text{at} \quad \zeta = \pm\tfrac{1}{2}. \tag{16}$$

The solution of (14) and (16) is

$$(P-1)F = \frac{\lambda\tanh\frac{1}{2}\lambda}{\alpha\tanh\frac{1}{2}\alpha}\left\{1 - \frac{\cosh\alpha\zeta}{\cosh\frac{1}{2}\alpha}\right\} - \left\{1 - \frac{\cosh\lambda\zeta}{\cosh\frac{1}{2}\lambda}\right\}. \tag{17}$$

At the walls the mean velocity $\bar{u}$ must be zero; we use these conditions along with (4), (12) and (15) and integrate (11) twice. We then integrate by parts to remove the double integral, and, as F is an even function, we find that the average mean velocity $\bar{\bar{u}}$ of the fluid is given by

$$\bar{\bar{u}} = \frac{-\kappa^2}{\nu k}\left(\frac{g\rho_{1w}}{\rho_0 U^2}\right)^2 \mathscr{I}\left[\int_0^{\frac{1}{2}} \zeta F \tilde{F}' \, d\zeta\right], \tag{18}$$

where a second overbar denotes a mean value with respect to z and $\mathscr{I}$ denotes the imaginary part. For large and small values of Ω, ΩP^{-1} we find that

$$\bar{\bar{u}} = \frac{k\kappa}{4P(P+1)}\left(\frac{g\rho_{1w}}{k\rho_0 U^2}\right)^2\left[1 - \frac{3P^2 + P^{\frac{3}{2}} + 10P + P^{\frac{1}{2}} + 3}{2^{\frac{1}{2}}\Omega^{\frac{1}{2}}(P^{\frac{1}{2}}+1)(P+1)} + O(\Omega^{-1})\right], \quad (\Omega \text{ large}); \tag{19}$$

$$\bar{\bar{u}} = \frac{k\kappa(P+1)}{12!\,P^4}\left(\frac{g\rho_{1w}}{k\rho_0 U^2}\right)^2 [\Omega^5 + O(\Omega^7)], \quad (\Omega \text{ small}). \tag{20}$$

The solution for F obtained by Stern is the limit of (17) when $\lambda \to 0$ (since $\lambda \sim \Omega^{\frac{1}{2}}$) with α fixed ($P \to 0$ with ΩP^{-1} fixed). Since P exceeds unity for almost all liquids, it follows that the value of $\bar{\bar{u}}$ given by Stern for large values of ΩP^{-1} but for a small value of Ω is not very relevant. We see from (19) that, for large values of the frequency, $\bar{\bar{u}} \propto U^{-4}$, whereas Stern gave $\bar{\bar{u}} \propto U^{-2\cdot 5}$. For low values of the frequency, (20) gives, for $P \to 0$, a result identical to the comparable one obtained by Stern, after correcting one of his integrals. Stern's theory is valid only if the diffusion time, $h^2/4\kappa$, required for heat to diffuse from the wall to the channel centre, is small compared with both the viscous diffusion time $h^2/4\nu$ and also the period $2\pi/\omega$. This requires $\omega h^2/\kappa = \Omega \ll 8\pi$ and $P \ll 1$, whereas in the experimental work of Fultz *et al.* and Stern P was about 6 (for water), and the smallest value of Ω was 1100.

4. The open problem

The experiments of Fultz *et al.* and Stern were carried out with an upper free surface rather than with the fluid contained between two plates. Thus, it seems worthwhile to study a mathematical model with an upper free surface, as in

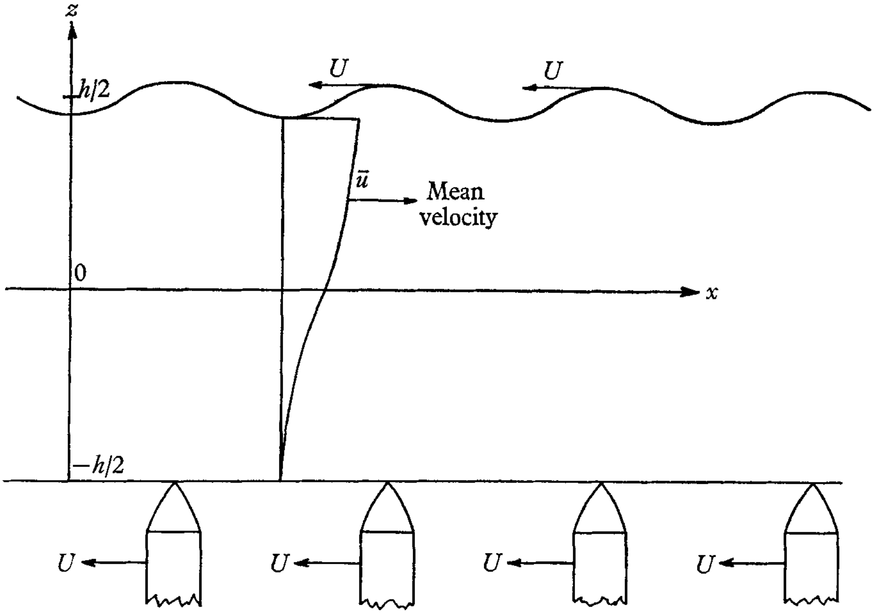

FIGURE 1. The motion with an upper free surface.

figure 1. For the steady temperature field, there will be a balance at the surface between the rate of transfer of heat by conduction in the liquid and the rate of loss of heat by convection and radiation. Under conditions similar to those in the experiments, with a depth of about 2 cm and a temperature range of 40–50 °C, the heat loss by radiation will be slightly greater than the convective heat loss, and the value of $(\partial\overline{T}/\partial\zeta)/\overline{T}$ will be about 0·3–0·4. It is thus not a happy approximation to suppose that $\overline{T}$ and ρ_0 are constant, although as the fluid is not deep the actual variations of $\overline{T}$ and ρ_0 throughout the fluid are small. We take $\overline{T}$, ρ_0 as constant then in order to present the simplest solution which nevertheless contains all the characteristics of the more general solution obtained with an upper boundary condition of the form $\partial\overline{T}/\partial\zeta+\beta\overline{T} = 0$. For the unsteady temperature field, there is no appreciable heat loss by convection, and nearly all the heat transfer is a balance between conduction and radiation. The true boundary condition is thus of the form $\partial T_1/\partial\zeta+\beta T_1 = 0$, where β has a value of about 0·2. However, for simplicity, we assume $\beta = 0$, so as to obtain definite results which indicate the nature of the solution. Indeed, our solution will be the leading term of the general solution expanded in powers of β.

Thus, we take $T_1' = 0$ at $\zeta = \frac{1}{2}$; the solution of (10) is then

$$\frac{T_1}{T_{1w}} = \frac{\cosh\lambda(\zeta-\frac{1}{2})}{\cosh\lambda}, \tag{21}$$

with $T_1 = T_{1w}$ at $\zeta = -\frac{1}{2}$. Hence, using (7), (12), (15), (21) and the equation of state, the equation for $F(\zeta)$ is now

$$F^{\text{iv}} - \alpha^2 F'' = \alpha^2\lambda^2 \frac{\cosh\lambda(\zeta-\frac{1}{2})}{\cosh\lambda}. \tag{22}$$

The boundary conditions at the plate are $u' = w' = 0$, so that $F = F' = 0$ when $\zeta = -\frac{1}{2}$. At the surface, the pressure is constant and equal to the atmospheric pressure (pressure fluctuations in the atmosphere caused by water waves are negligible) and there is zero stress across the surface.

Let the upper surface be at $z = \frac{1}{2}h + \eta(x,t)$ and suppose that

$$2\eta = \eta_0 e^{ik(x+Ut)} + \tilde{\eta}_0 e^{-ik(x+Ut)}, \tag{23}$$

with η_0 constant. Because the free surface moves with the fluid, we can write

$$\frac{D}{Dt}\{z - \tfrac{1}{2}h - \eta(x,t)\} \equiv -(U+u')\frac{\partial\eta}{\partial x} + w' = 0,$$

on using (4) and (23). Thus, as u'/U is of order ρ_{1w}/ρ_0, we have to the required order of approximation, near the upper surface,

$$w' = U\,\partial\eta/\partial x. \tag{24}$$

We obtain our first boundary condition at the free surface by considering the pressure there. This will be continuous so that

$$\rho_0 g\eta - p' + 2\mu\frac{\partial w'}{\partial z} = \text{constant}. \tag{25}$$

It is permissible to omit the higher-order terms due to the slope of the surface. Now differentiate (25) with respect to x, eliminate η using (24), and eliminate the pressure gradient by use of (1), retaining only those terms of order ρ_{1w}/ρ_0 to obtain

$$\nu\frac{\partial^2 u'}{\partial z^2} + 3\nu\frac{\partial^2 u'}{\partial x^2} - \frac{\partial u'}{\partial t} - \frac{gw'}{U} = 0 \quad (\text{at } z = \tfrac{1}{2}h),$$

where use has also been made of the continuity equation (3). We may ignore the second term as $k^2h^2 \ll 4\pi^2$. Using (4), (12) and (15), the required boundary condition may be expressed as

$$F''' - \alpha^2 F' + \alpha^2\Lambda F = 0 \quad \text{at} \quad \zeta = \tfrac{1}{2}. \tag{26}$$

The values of Λ in the experiments were about 135 (Fultz *et al.*), 7500 (Fultz *et al.*) and 50 (Stern).

We obtain our second boundary condition from the zero-stress requirement at the surface, ignoring any possible effects due to surface tension or evaporation. We determine this condition correct to *second* order in ρ_{1w}/ρ_0 because, although we only need terms of order ρ_{1w}/ρ_0 to find the boundary condition on F, we shall need the fuller condition later to enable us to solve the mean-motion equation. Small suffixes s and n are used, as in figure 2, to denote the tangential and normal directions at the surface, with ϵ denoting the small clockwise angle between the z-direction and the outward normal.

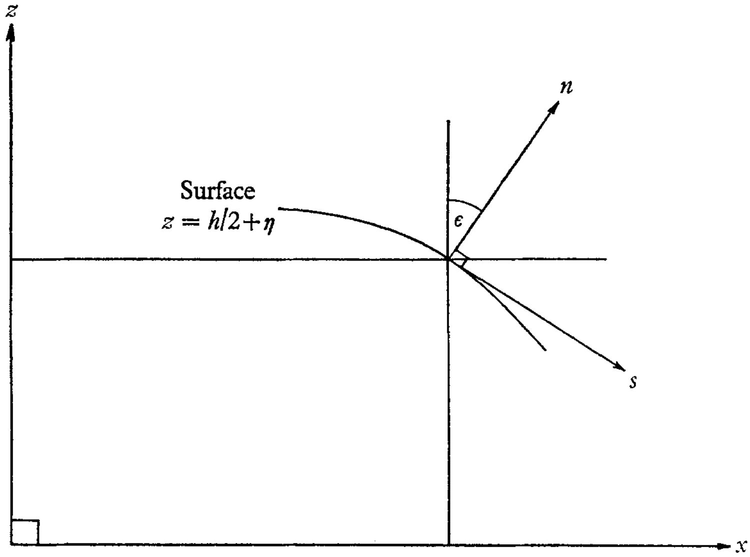

FIGURE 2. Co-ordinate directions at the upper surface.

At the upper surface there is zero shear stress so that

$$\nu\left(\frac{\partial u_s}{\partial n}+\frac{\partial u_n}{\partial s}\right) = 0 \quad \text{at} \quad z = \tfrac{1}{2}h+\eta. \tag{27}$$

For small values of ϵ, of order ρ_{1w}/ρ_0, we transform (27) to our original co-ordinates (x,z), ignoring terms of order $(\rho_{1w}/\rho_0)^3$. In this connexion, it must be remembered that u', w' are of order ρ_{1w}/ρ_0, but that $\bar{u}$ is of order $(\rho_{1w}/\rho_0)^2$. We find after neglecting a term $\partial w'/\partial x$ which is small compared with $\partial u'/\partial z$ when $k^2h^2 \ll 4\pi^2$ and using (3), that

$$\frac{\partial u'}{\partial z}+\frac{\partial \bar{u}}{\partial z}+4\epsilon\frac{\partial u'}{\partial x}+u'\frac{\partial \epsilon}{\partial x} = O(\rho_{1w}/\rho_0)^3 \quad \text{at} \quad z = \tfrac{1}{2}h+\eta.$$

Hence, after using Taylor's theorem to obtain the appropriate condition at $z = \frac{1}{2}h$ we have, to order (ρ_{1w}/ρ_0),^{2} that

$$\frac{\partial u'}{\partial z}+\frac{\partial \bar{u}}{\partial z}+\eta\frac{\partial^2 u'}{\partial z^2}+4\epsilon\frac{\partial u'}{\partial x} = 0 \quad \text{at} \quad z = \tfrac{1}{2}h. \tag{28}$$

Only the first term in (28) is of order ρ_{1w}/ρ_0, so that, with reference to (4) and (15), the required boundary condition on F is simply

$$F''' = 0 \quad \text{at} \quad \zeta = \tfrac{1}{2}. \tag{29}$$

The function $F(\zeta)$ may now be found from (22), imposing $F = F' = 0$ at $\zeta = -\frac{1}{2}$ and the conditions (26) and (29). The solution is

$$F = A + B\zeta + C\cosh\alpha\zeta + D\sinh\alpha\zeta + \frac{\cosh\lambda(\zeta-\frac{1}{2})}{(P-1)\cosh\lambda}, \tag{30}$$

where A, B, C, D are rather cumbersome functions of α, λ, P (or λ/α) (which may be obtained from the boundary conditions as the solution of four simultaneous linear equations).

To determine the mean motion, we integrate (11) once so that

$$\nu\frac{d\bar{u}}{dz} - \overline{u'w'} = c \quad \text{(a constant).} \tag{31}$$

Now $\epsilon = -\partial\eta/\partial x$ and, with the aid of (4) and (15), we may show that the last two terms in (28) may be ignored compared with $\eta\,\partial^2u'/\partial z^2$ because $k^2h^2 \ll 4\pi^2$. Thus, if we make this approximation and extract the mean part of (28) of order $(\rho_{1w}/\rho_0)^2$, we find that

$$\frac{d\bar{u}}{dz} + \left(\overline{\eta\frac{\partial^2 u'}{\partial z^2}}\right) = 0, \quad \text{at} \quad \zeta = \tfrac{1}{2}. \tag{32}$$

This enables us to find the value of c in (31) as follows: set $\zeta = \frac{1}{2}$ and eliminate $\bar{u}$ between (31) and (32); then use (4), (24) to express c in terms of w_1 and its derivatives. We may then use (12) and (15) to express c in terms of F and its derivatives, and the resulting expression may be simplified by using (26) to give simply $c = 0$. Thus, as expected, the fluid imparts no energy to the atmosphere, and the mean motion is governed entirely by the Reynolds stress.

We again use $\bar{u}$ to denote the average mean velocity of the fluid with respect to z. At the plate $\bar{u} = 0$; we use this condition, along with $c = 0$, to integrate (31) twice. After a single integration by parts and use of (4), (12) and (15), we obtain

$$\bar{u} = \frac{-\kappa^2}{2\nu k}\left(\frac{g\rho_{1w}}{\rho_0 U^2}\right)^2 \mathscr{I}\left[\int_{-\frac{1}{2}}^{\frac{1}{2}} (\zeta-\tfrac{1}{2})F\tilde{F}'\,d\zeta\right]. \tag{33}$$

For large and small values of Ω, ΩP^{-1} we find that

$$\bar{u} = \frac{k\kappa}{4P(P+1)}\left(\frac{g\rho_{1w}}{k\rho_0 U^2}\right)^2\left[1 - \frac{3P^2+P^{\frac{3}{2}}+10P+P^{\frac{1}{2}}+3}{2^{\frac{1}{2}}\Omega^{\frac{1}{2}}(P^{\frac{1}{2}}+1)(P+1)} - \frac{2^{\frac{1}{2}}(P+P^{\frac{1}{2}}+2)}{\Omega^{\frac{1}{2}}(P^{\frac{1}{2}}+1)}\frac{\Lambda}{(\Lambda-1)} + O(\Omega^{-1})\right] \quad (\Omega\ \text{large}); \tag{34}$$

$$\bar{u} = \frac{k\kappa}{2.6!P^4}\left(\frac{g\rho_{1w}}{k\rho_0 U^2}\right)^2\left[\frac{13P+90}{1232} - \frac{3\Lambda}{64}\right][\Omega^5 + O(\Omega^7)] \quad (\Omega\ \text{small}). \tag{35}$$

In figure 3 we show how the phase angles between u', w' and T' vary with depth for a typical large frequency case. We note that in the upper half of the fluid u', w' are almost completely out of phase, whereas the phase angle between u', T' is continuously changing due to the high speed of the thermal field compared to the speed of heat transport.

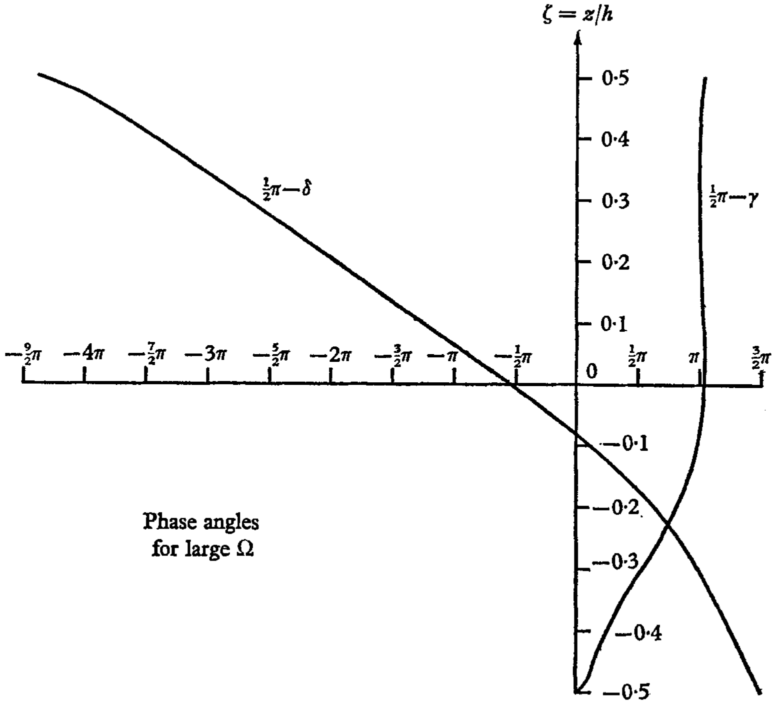

FIGURE 3. Typical variation of phase angles between u', w' and T' with depth for large values of Ω, ΩP^{-1}; (results for $P = 6$, $\Lambda = 1000$; $\Omega = 1000$). γ is the phase lead of u' over w' and δ is the phase lead of u' over T'.

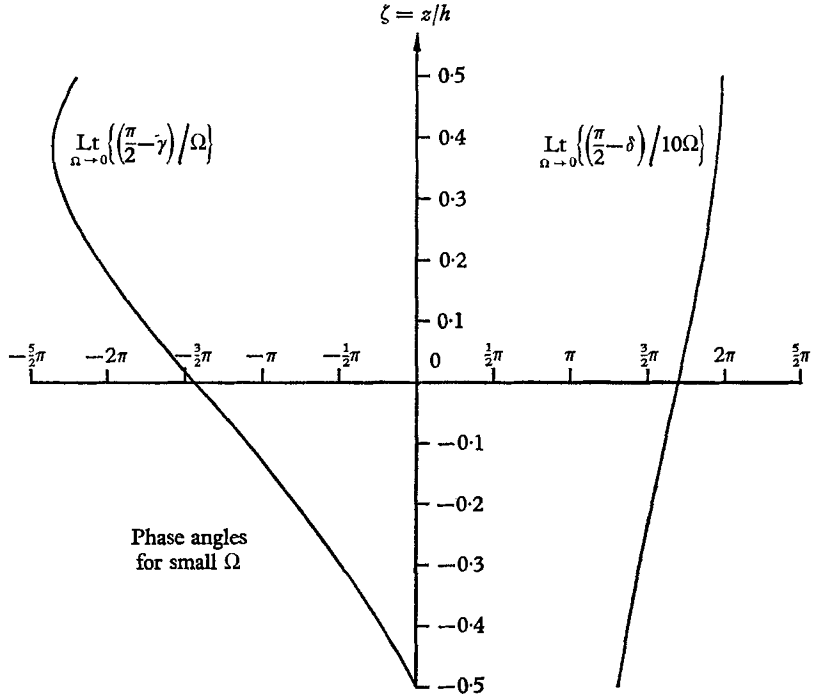

FIGURE 4. Variation of phase angles between u', w' and T' with depth for small values of Ω, ΩP^{-1}; (results for $P = 6$, $\Lambda = 1000$). γ is the phase lead of u' over w' and δ is the phase lead of u' over T'.

In figure 4 we show how the phase angles between u', w' and T' vary with depth when the frequency is small but Λ is so large that the mean motion is in the same direction as that of the imposed thermal field. The phase angles are of order Ω and we have plotted the limiting values of the phase angles divided by Ω as Ω tends to zero. We note that the phase angles are almost linear functions of the depth except close to the upper free surface.

In figure 5 we indicate the variation with depth of the mean motion $\overline{u}$ for both the high- and the low-frequency cases. In the high-frequency case we obtain a boundary-layer type profile with zero gradient at both the boundaries, the flow is in the opposite direction to that of the imposed thermal field (which travels in the negative x-direction). In the low-frequency case which is dominated by the large value of Λ, the mean motion is in the opposite direction and has quite a different profile.

In the experiments the flow was observed by means of marked particles on the free surface whose drift velocity could be measured. Our analysis may be compared with the experimental results, by considering the mass-transport velocity $\overline{U}$, which is the steady drift velocity of the fluid particles. It is known (Longuet-Higgins 1953) that, when the motion is a fluctuation from a state of rest, the mass-transport velocity is given to second order in ρ_{1w}/ρ_0 by

$$\overline{U} = \overline{u} + \overline{\left(\frac{\partial u'}{\partial x}\int u'\,dt + \frac{\partial u'}{\partial z}\int w'\,dt\right)}. \tag{36}$$

In (36) the integrals are indefinite, and the overbar denotes the mean value with respect to time over a complete period; there is no mass transport in the z-direction. At the surface $\partial u'/\partial z = 0$, to order ρ_{1w}/ρ_0, from (28) and (29). Thus we may use (4), (15) and (31), integrated once with $c = 0$, to write (36) at the surface in the form

$$\overline{U}_s = \frac{k\kappa}{2P}\left(\frac{g\rho_{1w}}{k\rho_0 U^2}\right)^2\left[\mathscr{I}\left\{\int_{-\frac{1}{2}}^{\frac{1}{2}} F\tilde{F}'\,d\zeta\right\} - \frac{P}{\Omega}(F'\tilde{F}')_{\frac{1}{2}}\right]. \tag{37}$$

If we measure $\overline{U}_s$ experimentally, we may obtain a good estimate of $\overline{\overline{u}}$, for we may use (34), (35) and (37) to obtain

$$\frac{\overline{\overline{u}}}{\overline{U}_s} = 1 - \frac{3P^2 + P^{\frac{3}{2}} + 10P + P^{\frac{1}{2}} + 3}{2^{\frac{1}{2}}\,\Omega^{\frac{1}{2}}\,(P^{\frac{1}{2}}+1)\,(P+1)} + O(\Omega^{-1}) \quad (\Omega \text{ large}); \tag{38}$$

$$\frac{\overline{\overline{u}}}{\overline{U}_s} = -\frac{\Omega^2}{20P^2}\left\{\frac{13P+90}{1232} - \frac{3\Lambda}{64}\right\} + O(\Omega^4) \quad (\Omega \text{ small}). \tag{39}$$

It is also interesting to know the relationship between $\overline{u}_s$ and $\overline{U}_s$. Again we integrate (31) with $c = 0$ and use (4) and (15) to obtain $\overline{u}_s$, which is, in fact, the first term of (37). With this knowledge, and (37), we find that

$$\frac{\overline{u}_s}{\overline{U}_s} = 1 + \frac{2P(P+1)}{\Omega(P^{\frac{1}{2}}+1)^2}\left\{\frac{\Lambda}{\Lambda-1}\right\}^2 + O(\Omega^{-\frac{3}{2}}) \quad (\Omega \text{ large}); \tag{40}$$

$$\frac{\overline{u}_s}{\overline{U}_s} = -\frac{\Omega^2(11P+91-60\Lambda)}{5600P^2} + O(\Omega^4) \quad (\Omega \text{ small}). \tag{41}$$

With caution we note from (39) and (41) that at low frequencies and when Λ is also sufficiently small then $\overline{U}_s$ is opposite in sign to both $\bar{\bar{u}}$ and $\bar{u}_s$. Thus in this extreme case an experimental measure of the drift velocity is not a good guide to the mean motion; the second term of (36) is dominant.

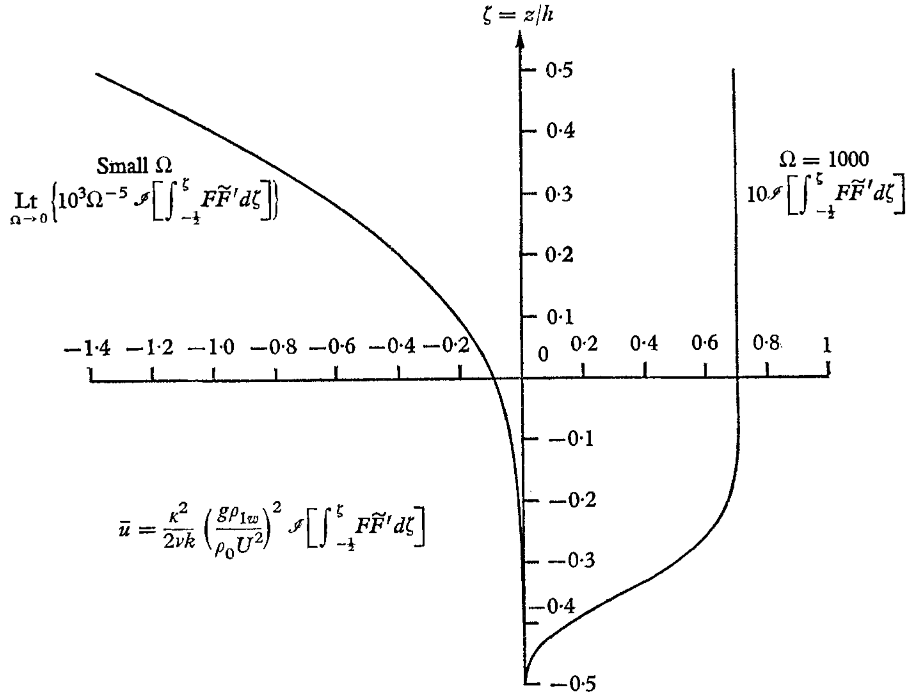

FIGURE 5. The variation with depth of the mean motion $\bar{u}$; (results for $P = 6$, $\Lambda = 1000$)

5. Discussion of the results

In the type of problem discussed in this paper, in which the fluid motion is determined solely by the temperature at the boundary, there are nevertheless several different physical mechanisms which decide the extent of the flow. These mechanisms are associated with the following velocities namely: (i) the velocity of propagation of a typical thermal conduction wave, (ii) the velocity of propagation of viscous diffusion, (iii) the velocity produced by buoyancy due to the thermal wave, and (iv) the velocity of the imposed temperature field at the boundary.

The most important point is that there is a phase lag, increasing with height, between the thermal conduction wave and the temperature field at the boundary. Only at very low frequencies is it realistic to assume infinite vertical conduction because then the vertical, buoyant convection takes place much more rapidly than horizontal heat transport. At high frequencies, as in the experiments, the velocity correlation is due almost entirely to thermal effects and not to viscosity. In particular, $\overline{u'w'}$ is not zero in the limit as $\nu \to 0$, $\kappa \neq 0$. The mean horizontal momentum of the flow depends on the square of the buoyancy force, which produces opposite velocities near the upper and lower boundaries. In the closed problem, there are two fixed boundaries on which these motions can produce horizontal stresses. At high frequencies, the flow is of a boundary-layer character of thickness $(\kappa/\omega)^{\frac{1}{2}}$, outside of which the mean velocity is almost uniform.

For the closed problem, we see from (19), that at high frequencies the value of $\bar{\bar{u}}$ is positive and proportional to U^{-4}. At low frequencies (20) indicates that $\bar{\bar{u}}$ is again positive and proportional to U. It seems reasonable to suppose then that at all frequencies the net mean momentum is in the opposite direction to that of the thermal field, (which travels in the *negative* x-direction), with a maximum value occurring for some intermediate value of U. For the open problem, (34) indicates that $\bar{\bar{u}}$ is again positive at high frequencies provided $(1-\Lambda)$ is not small.† The value of $(1-\Lambda)$ was far from small in all the experimental work, details of which are given in table 1. At low frequencies, with reference to (35), the sign of $\bar{\bar{u}}$ is the same as the sign of $(52P+360-231\Lambda)$. If the channel is shallow enough (Λ small), then $\bar{\bar{u}}$ will be positive as before. However, with, say, $P = 6$, the mean motion will be in the same direction as that of the thermal field for $\Lambda > 3$. All the experiments were high frequency cases, and they confirmed that the motion is in the direction opposite to that of the thermal field. Unfortunately no detailed velocity measurements were taken. We note from (38) that the mean velocity must have a boundary-layer profile at high frequencies.

	Ω	P	Λ	kh	$k\kappa$
Stern	1100	4·1	50	0·133	$1{\cdot}01\times10^{-4}$
Fultz *et al.*	1300	5·5	7500	0·382	$0{\cdot}99\times10^{-4}$
Fultz *et al.*	9900	6·0	135	0·382	$0{\cdot}97\times10^{-4}$

TABLE 1. Experimental data

The energy equation for the velocity fluctuations may be obtained as follows: take the linearized forms of (1) and (2), and multiply, respectively, by u' and w', and add to obtain

$$\frac{1}{2}\frac{\partial}{\partial t}(u'^2+w'^2) = -\frac{1}{\rho_0}\left(u'\frac{\partial p'}{\partial x}+w'\frac{\partial p'}{\partial z}\right)-w'\frac{\rho'}{\rho_0}g+\nu(u'\nabla^2u'+w'\nabla^2w'). \qquad (42)$$

Now integrate over a volume of the fluid to obtain

$$\frac{\partial}{\partial t}\iint \tfrac{1}{2}(u'^2+w'^2)\,dx\,dz = -\iint w'\frac{\rho'}{\rho_0}g\,dx\,dz-\nu\iint \omega'^2\,dx\,dz, \qquad (43)$$

where ω' is the fluctuating vorticity; the pressure terms give no contribution. Thus we see that the energy of the motion derives from work done by the gravitational field due to density variations caused by the thermal field. A balance is then maintained with viscous dissipation.

† When $\Lambda = 1$ the thermal field moves at the same speed as free surface waves and resonance occurs. Thus the surface waves continuously extract energy from the flow and increase in amplitude until a breakdown occurs. Unfortunately no experimental work has as yet been done for values of Λ near unity.

6. Concluding remarks

In this paper we have primarily examined the equilibrium motion developed when heat sources move under a liquid layer which has an upper free surface. We have found that a net mean momentum is present, indicating that the velocity fluctuations transfer momentum which is balanced by the stress developed by the mean velocity field. We have not considered how this mean momentum is developed from rest, but Stern has suggested the following sequence: initially the mean momentum flux is zero, but a Reynolds stress gradually generates momentum flux in a region far away from the constraining boundary; this is countered by a momentum flux in the opposite direction near the boundary. The boundary exercises a restraint and destroys this neighbouring momentum but leaves that which is far away from the boundary. Thus, in equilibrium, a mean momentum flux persists which is far away from the boundary, where the stress is zero.

We have found that the mean momentum flux and the mass-transport velocity at the surface are directed oppositely to the travelling thermal field, at least under certain conditions. These are (i) that the wavelength is large compared with the depth of the fluid and (ii) that the frequency parameter is sufficiently high. Both of these conditions were realized in all the experiments so that agreement is established. Condition (ii) may be relaxed for the closed problem of §3.

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